– Analiza danych niepewnych –

Introduction to Fuzzy Statistics

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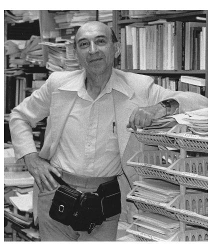
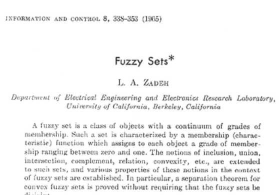
Outline

Introduction



Fuzzy sets were introduced in 1965 by Lofti Zadeh (1921-2017), the father of fuzzy mathematics and fuzzy logic.

Since the seminal articles in 1965 (*Fuzzy sets* and *Fuzzy sets and systems*), research and applications based on Zadeh's work have rapidly increased.



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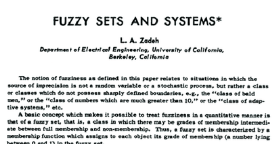
Fuzzy Sets*

L. A. Zadeh

Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.



FUZZY SETS AND SYSTEMS*

L. A. Zadeh

Department of Electrical Engineering, University of California, Berkeley, California

The notion of fuzziness as defined in this paper relates to situations in which the source of imprecision is not a random variable or a stochastic process, but rather a class or classes which do not possess sharply defined boundaries, e.g., the "class of bald men," or the "class of numbers which are much greater than 10," or the "class of adaptive systems," etc.

A basic concept which makes it possible to treat fuzziness in a quantitative manner is that of a fuzzy set, that is, a class in which there may be grades of membership intermediate between full membership and non-membership. Thus, a fuzzy set is characterized by a membership function which assigns to each object its grade of membership (a number lying between 0 and 1) in the fuzzy set.

Introduction

By and large, fuzzy set theory is a mathematical approach dealing with ambiguousness, subjectiveness, and imprecision.

Some of the disciplines revolving around fuzzy set theory:

- fuzzy topology, geometry and algebra
- fuzzy logic
- fuzzy numerical analysis and fuzzy differential equations
- fuzzy statistics and data analysis
- control theory and intelligent systems, fuzzy optimization
- artificial intelligence, approximate reasoning
- soft computing, knowledge based systems

What is a fuzzy set?
Fuzzy sets generalize classic crisp sets

A **fuzzy subset** $\tilde{A}$ of a reference set $\Omega \subset \mathbb{R}$ is defined by means of its characteristic function:

$$\xi_{\tilde{A}} : \Omega \to [0,1]$$

In general, for upper semicontinuous functions $\xi_{\tilde{A}}$, α-level sets

$$[\xi_{\tilde{A}}]_\alpha = \left\{x \in \mathbb{R} : \xi_{\tilde{A}}(x) \geq \alpha, \alpha \in (0,1]\right\}$$

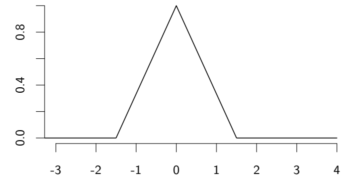
can be constructed as nonempty and compact convex sets. This allows the **generalization** of interval-valued calculus to these objects.

Several definitions of $\xi_{\tilde{A}}$ are available (e.g., triangular, gaussian, trapezoidal).

What is a fuzzy set?
Fuzzy numbers as special fuzzy sets

Triangular

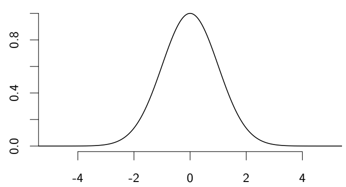
$$\text{tfn}(x; x_0, x_l, x_u) = \begin{cases} 0, & \text{if } x < x_l \text{ or } x > x_u \\ 1, & \text{if } x = x_0 \\ \frac{(x_0 - x)}{(x_0 - x_l)} & \text{if } x \in [x_l, x_0) \\ \frac{(x - x_0)}{(x_r - x_0)} & \text{if } x \in (x_0, x_l] \end{cases}$$



What is a fuzzy set?
Fuzzy numbers as special fuzzy sets

Gaussian

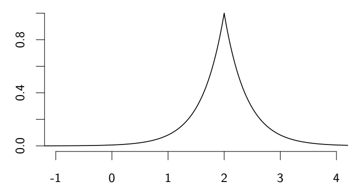
$$\text{gfn}(x; x_0, s_l, s_r) = \begin{cases} \exp\left(-\frac{1}{2s_l^2}(x - x_0)^2\right) & \text{if } x \in (-\infty, x_0) \\ \exp\left(-\frac{1}{2s_r^2}(x - x_0)^2\right) & \text{if } x \in [x_0, \infty) \end{cases}$$



What is a fuzzy set?
Fuzzy numbers as special fuzzy sets

Exponential

$$\text{efn}(x; x_0, t_l, t_r) = \begin{cases} \exp\left(-\frac{1}{t_l}(x - x_0)\right) & \text{if } x \in (-\infty, x_0) \\ \exp\left(-\frac{1}{t_r}(x - x_0)\right) & \text{if } x \in [x_0, \infty) \end{cases}$$

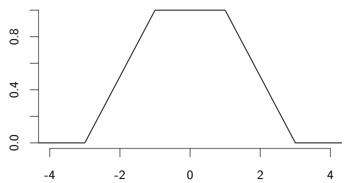


What is a fuzzy set?

Fuzzy numbers as special fuzzy sets

Trapezoidal

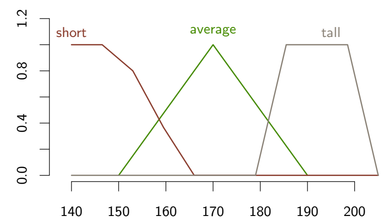
$$\text{tfn}(x; x_0, x_1 x_l, x_r) = \begin{cases} 0, & \text{if } x < x_l \text{ or } x > x_u \\ 1, & \text{if } x \in [x_0, x_1] \\ \frac{(x_0 - x)}{(x_0 - x_l)} & \text{if } x \in [x_l, x_0) \\ \frac{(x - x_0)}{(x_r - x_0)} & \text{if } x \in (x_0, x_l] \end{cases}$$



What is a fuzzy set?

Fuzzy sets generalize classic crisp sets

An example modeling the person's height (short, average, tall) with a **fuzzy variable** (i.e., a collection of fuzzy sets):



What is a fuzzy set?

Fuzzy sets generalize classic crisp sets

The characteristic function $\xi_{\tilde{A}} : \Omega \to [0, 1]$ provides information about the degree to which elements (or intervals) of Ω belong to $\tilde{A}$ (i.e., larger values denote higher degrees of set membership).

The use of $\xi_{\tilde{A}}$ to represent $\tilde{A}$ is usually said **vertical representation**.

Another way to define $\tilde{A}$ is by adopting a **horizontal representation**:

$$\forall \alpha \in [0, 1] : \quad [\xi_{\tilde{A}}]_\alpha = \{x \in \Omega : \xi_{\tilde{A}} \geq \alpha\}$$

$([\xi_{\tilde{A}}]_\alpha)_{\alpha > 0}$ is a collection of *slices* of $\xi_{\tilde{A}}$ called α-cuts (they are crisp sets).

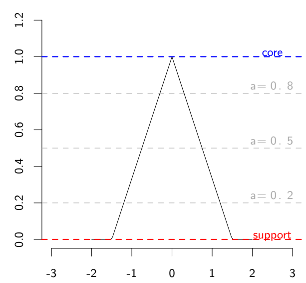
It follows that:

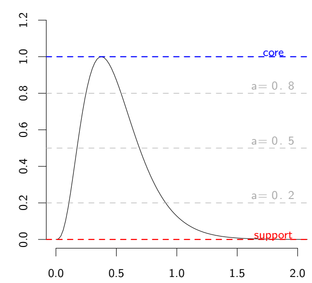
$[\xi_{\tilde{A}}]_{\alpha=0} = \Omega$

$[\xi_{\tilde{A}}]_\alpha \supseteq [\xi_{\tilde{A}}]_{\alpha'} \quad \alpha < \alpha'$

What is a fuzzy set?

Fuzzy sets generalize classic crisp sets

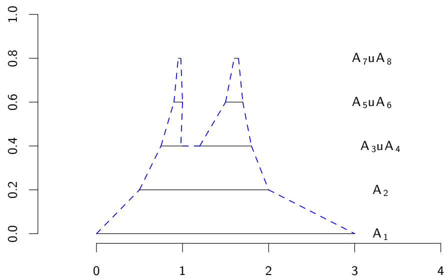




Support of $\tilde{A}$: $[\xi_{\tilde{A}}]_{\alpha > 0}$

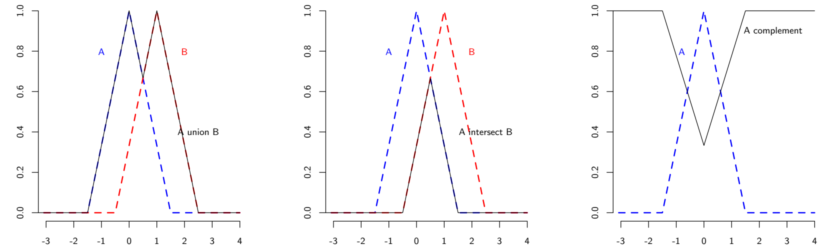
Core of $\tilde{A}$: $[\xi_{\tilde{A}}]_{\alpha = 1}$

What is a fuzzy set?
Fuzzy sets generalize classic crisp sets



The vertical representation can also be obtained via the horizontal one:
$\xi_{\tilde{A}}$ is the **upper envelope** of $([\xi_{\tilde{A}}]_\alpha)_{\alpha \in \mathcal{L}}$.

What is a fuzzy set?
Basic operations between fuzzy sets

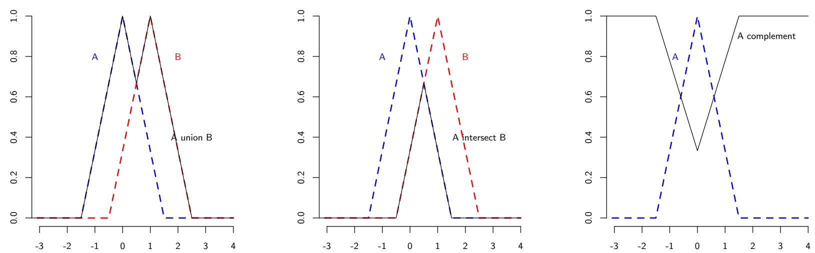


Union $\xi_{\tilde{A}\cup\tilde{B}}(x) = \max\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(x)\}$

Intersection $\xi_{\tilde{A}\cap\tilde{B}}(x) = \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(x)\}$

Complement $\overline{\xi}_{\tilde{A}}(x) = 1 - \xi_{\tilde{A}}(x)$

What is a fuzzy set?
Basic operations between fuzzy sets



Note: The operations of intersection and union can also be defined in terms of **T-norm** (eg: product, Lukasiewicz) and **T-conorm** (eg: maximum, bounded sum).

Calculus with fuzzy numbers
The extension principle

To extend the most elementary operations (E) of **addition/subtraction**, **multiplication**, and **division** to handle with fuzzy numbers, the *Zadeh's extension principle* (EP) could be applied in this case:

$$\xi_{\tilde{c}}(z) \quad = \sup_{z=E(x,y)} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

Although it produces reasonable results in most applications, this generalization does not guarantee that $\xi_{\tilde{c}}$ is still a fuzzy number.

Alternatives exist and they are based on generalization of the EP, use of shape/reference functions, and reduction to interval calculus.

Calculus with fuzzy numbers
Fuzzy numbers

Among the various type of fuzzy sets, those which are defined over $\mathbb{R}$ are of particular importance.

A fuzzy set $\tilde{X}$ is a **fuzzy number** $\tilde{x}$ if it satisfies the following conditions:

- $\tilde{X}$ is normal (i.e., its maximum degree of membership is one)
- $\tilde{X}$ is convex
- $\xi_{\tilde{X}}$ is at least piecewise continuous

A fuzzy number $\tilde{x}$ is *positive* if $[\xi_{\tilde{x}}]_{\alpha=0} \subseteq (0,\infty)$, *negative* if $[\xi_{\tilde{x}}]_{\alpha=0} \subseteq (-\infty,0)$.

The set of fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

Calculus with fuzzy numbers
LR fuzzy numbers

Dubois and Prade's **LR fuzzy numbers** consist of re-parameterizing $\xi_{\tilde{c}}$ in terms of two monotonic decreasing and left-continuous *shape functions*:

$$L : \mathbb{R}^+ \to [0,1] \qquad R : \mathbb{R}^+ \to [0,1]$$

with

$$L/R(\upsilon) \begin{cases} = 0 & \text{if } \upsilon = 1 \\ = 1 & \text{if } \upsilon = 0 \\ > 0 & \text{if } \upsilon < 1 \\ < 1 & \text{if } \upsilon > 0 \end{cases}$$

and where

$$\xi_{\tilde{c}}(x) = \begin{cases} L\left(\frac{m-x}{l}\right) & \text{if } x < m \\ R\left(\frac{x-m}{r}\right) & \text{if } x \geq m \end{cases}$$

with m, l, r being the **mode**, **left/right spread** ($l > 0$,$r > 0$).

Calculus with fuzzy numbers
LR fuzzy numbers

For instance, with the LR parametrization the **triangular fuzzy number**:

$$\xi_{\tilde{c}}(x) = \begin{cases} L\left(\frac{x_0-x}{x_0-x_l}\right) & \text{if } x < x_0 \\ R\left(\frac{x-x_0}{x_u-x_0}\right) & \text{if } x \geq x_0 \end{cases}$$

with

$$L(u) = R(u) = \max\{0, 1-u\}$$

Calculus with fuzzy numbers
LR fuzzy numbers

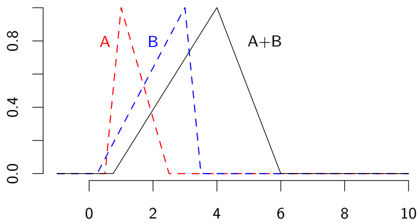
Given $\tilde{a}$ and $\tilde{b}$, in order to compute $\tilde{c} = E(\tilde{a},\tilde{b})$ - with $E()$ being one of the basic operation - it is needed that $E()$ still produces LR-fuzzy number (i.e., *closure*). This requires some restrictions on the type of L/R shape functions being involved.

To ensures closureness of $E()$, L/R functions need to be approximated in some cases (e.g., via secant or tangent techniques). This is especially valid for *multiplication* of fuzzy numbers [24].

Addition (triangular case)

$$\tilde{c} = \tilde{a} + \tilde{b}$$

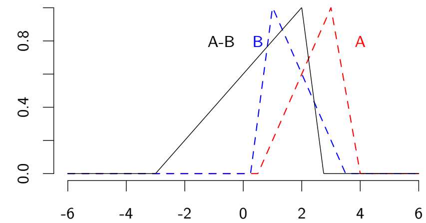
$$(m_a + m_b, l_a + l_b, r_a + r_b)_{LR} = (m_a, l_a, r_a)_{LR} + (m_b, l_b, r_b)_{LR}$$



Subtraction (triangular case)

$$\tilde{c} = \tilde{a} - \tilde{b}$$

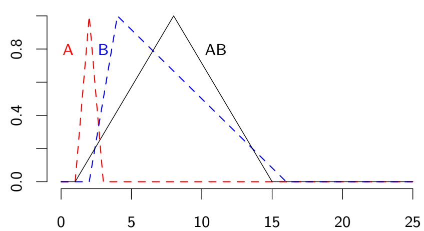
$$(m_a - m_b, l_a + r_b, r_a + l_b)_{LR} = (m_a, l_a, r_a)_{LR} - (m_b, l_b, r_b)_{LR}$$



Multiplication (triangular positive case)

$$\tilde{c} = \tilde{a} \cdot \tilde{b}$$

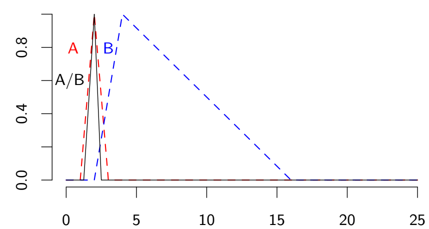
$$(m_a m_b, \underbrace{m_a l_b + m_b l_a - l_a l_b, m_a r_b + m_b r_a + r_a r_b}_{\text{secant approximation}})_{LR} = (m_a, l_a, r_a)_{LR} \cdot (m_b, l_b, r_b)_{LR}$$



Division (triangular positive case)

$$\tilde{c} = \tilde{a} \cdot \tilde{b}^{-1}$$

$$(m_a/m_b, \underbrace{(x_a r_b + x_b l_a) m_b^{-2}, (x_a l_b + x_b r_a) m_b^{-2}}_{\text{tangent approximation}})_{LR} = (m_a, l_a, r_a)_{LR}/(m_b, l_b, r_b)_{LR}$$



Calculus with fuzzy numbers
Interval calculus

To overcome the difficulties of the calculus based on L-R shape functions, one can turn to the decomposition of a $\tilde{x}$ as a (finite) sequence of α intervals $([\xi_{\tilde{x}}]_\alpha)_{\alpha \in \mathcal{L}}$. In this case,

$$\tilde{c} = E(\tilde{a}, \tilde{b})$$
$$C_\alpha = E(A_\alpha, B_\alpha)$$
$$[l_a, r_a]_\alpha = E([l_a, r_a]_\alpha, [r_a, r_b]_\alpha)$$

and the operation $E()$ is applied element-wise on the α-cuts of the fuzzy numbers being involved.

Calculus with fuzzy numbers
Interval calculus

Input: $\tilde{a}, \tilde{b}$ Output: $\tilde{c} = E(\tilde{a}, \tilde{b})$ using α-cuts of inputs

- **addition**
$$[l_a + r_a, l_b + r_b]_\alpha = [l_a, r_a]_\alpha + [l_b, r_b]_\alpha$$
- **subtraction**
$$[l_a - r_b, l_b - r_a]_\alpha = [l_a, r_a]_\alpha - [l_b, r_b]_\alpha$$
- **multiplication**
$$[\min(S_\alpha), \max(S_\alpha)]_\alpha = [l_a, r_a]_\alpha \cdot [l_b, r_b]_\alpha$$
$$S_\alpha = \{l_a l_b, l_a r_b, r_a l_b, r_a r_b\}$$
- **division**
$$[\min(S_\alpha), \max(S_\alpha)]_\alpha = [l_a, r_a]_\alpha / [l_b, r_b]_\alpha$$
$$S_\alpha = \{l_a/l_b, l_a/r_b, r_a/l_b, r_a/r_b\}$$
$$0 \notin [l_b, r_b]_\alpha$$

Calculus with fuzzy numbers
Semilinear spaces

In some cases, the addition/subtraction property does not hold here [10]:

$$A_\alpha + (-1 \cdot B_\alpha) + B_\alpha \neq A_\alpha$$

Thus, $(\mathcal{F}(\mathbb{R}), +, \cdot)$ is a **semilinear space**.

The *lack of subtraction* requires the generalization of the classical differentiation that fit the semilinear environment. The most successful approach implies the use of **Hukuhara differentiation** (e.g., see [16]):

$$C_\alpha = A_\alpha -_H B_\alpha \quad \text{such that } A_\alpha + B_\alpha = C_\alpha$$

Note that $-_H$ does not always exist (but if it does it is unique).

Calculus with fuzzy numbers
Semilinear metric spaces

The semilinear space $(\mathcal{F}(\mathbb{R}), +, \cdot)$ can be endowed with a **metric** which can be useful in statistical applications:

$$D_\tau^\lambda(A_\alpha, B_\alpha) = \left(\int_0^1 \left((\text{mid}A_\alpha - \text{mid}B_\alpha)^2 + \tau(\text{spr}A_\alpha - \text{spr}B_\alpha)^2 \right) \, d\lambda(\alpha) \right)^{\frac{1}{2}}$$

where $\text{mid}X_\alpha = (l_{x_\alpha} - r_{x_\alpha})/2$ and $\text{spr}X_\alpha = (r_{x_\alpha} - l_{x_\alpha})/2$
with λ being the Lebesgue measure.

A particular case of distance:

$$D^2(A_\alpha, B_\alpha) = (1-\tau)(\text{mid}A_\alpha - \text{mid}B_\alpha)^2 + \tau(\text{spr}A_\alpha - \text{spr}B_\alpha)^2$$

Outline

An elementary example

While probability formalizes the **randomness** of an aleatory experiment (i.e., uncertainty about the occurrence of random events), possibility formalizes the **epistemic imprecision** of a random experiment (i.e., uncertainty about the way random events are defined).

a proposition is uncertain if it involves a stochastic process; [...] an exact proposition may be uncertain ("it will be 4°C tomorrow"), and a proposition which is completely certain may be linguistically inexact ("it is warm now") [38]

An elementary example
Tossing a coin

Consider the experiment of tossing a (fair) coin with two possible outcomes: HEAD or TAIL. Over a sequence of independent tosses, the observer registered on a sheet whether the coin is HEAD (H) or TAIL (T).

$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	...
				T?	H?	H?	...

For certain tosses (i.e., $i \geq 5$) the observer got distracted and she/he was uncertain about the outcome (T or H ?).

An elementary example
Tossing a coin

Then, she/he decided to report the outcomes linguistically:

$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	...
				approx T	approx H	approx H	...

An elementary example
Tossing a coin

The coin tossing experiment can be usually modeled using a Bernoulli random variable $X \sim \mathcal{B}ern(x; \pi)$, with $\text{supp}(X) = \{0, 1\}$ and

$$X = 0 : \text{coin is HEAD} \qquad X = 1 : \text{coin is TAIL}$$

However, to deal with fuzzy outcomes (*was it head or tail?*), $\text{supp}(X)$ needs to be defined in terms of fuzzy sets

$$\text{supp}(X) = \{\xi_{\tilde{0}}, \xi_{\tilde{1}}\}$$

Abusing notation slightly, the original probability space

$$(\{\xi_{\tilde{0}}, \xi_{\tilde{1}}\}, \mathbb{P}_X)$$

enlarges to cope with the new source of uncertainty (fuzzy probability space).

An elementary example
Tossing a coin

$$(\overbrace{\{\xi_{\tilde{0}}, \xi_{\tilde{1}}\}}^{\text{fuzziness}}, \underbrace{\mathbb{P}_X}_{\text{randomness}})$$

In this case, the observer is **uncertain** about what

- the events 0 (H) and 1 (T) refer to (*fuzziness*)
- the occurrence of such events over a repeated experiment (*randomness*)

An elementary example
Tossing a coin

Let

$$\xi_0(x) = \begin{cases} 1.0 & x = 0 \\ 0.0 & x = 1 \end{cases} \qquad \xi_1(x) = \begin{cases} 0.0 & x = 0 \\ 1.0 & x = 1 \end{cases}$$

be the *crisp sets* for the events $X = 0$ (H) and $X = 1$ (T).

$$\xi_{\tilde{0}}(x) = \begin{cases} 0.9 & x = 0 \\ 0.2 & x = 1 \end{cases} \qquad \xi_{\tilde{1}}(x) = \begin{cases} 0.1 & x = 0 \\ 0.8 & x = 1 \end{cases}$$

be the *fuzzy sets* for the events $X \stackrel{\sim}{=} 0$ (approx H) and $X \stackrel{\sim}{=} 1$ (approx T).

An elementary example
Tossing a coin

In the Bernoulli case, $\mathbb{P}_X(x) = \pi^x(1-\pi)^{1-x}$ with $\pi \in (0, 1)$ as usual.

The probability of an event is equal to (e.g., see [3]):

$$\mathbb{P}_X(X = x) = \sum_{x \in \{0,1\}} \xi_{\tilde{x}}(x)\mathbb{P}_X(x)$$

where the Bernoulli probabilities are weighted by the membership function of the fuzzy sets.

An elementary example
Tossing a coin

$i = 1$	$i = 2$	$i = 3$	...
			...

With $\pi = 0.5$,

the probability of HEAD is

$$\begin{aligned}\mathbb{P}_X(X = 0) &= 1 \cdot (0.5^0 \cdot (1 - 0.5)^1) + 0 \cdot (0.5^1 \cdot (1 - 0.5)^0) \\ &= 0.50\end{aligned}$$

whereas the probability of TAIL is

$$\begin{aligned}\mathbb{P}_X(X = 1) &= 0 \cdot (0.5^0 \cdot (1 - 0.5)^1) + 1 \cdot (0.5^1 \cdot (1 - 0.5)^0) \\ &= 0.50\end{aligned}$$

An elementary example
Tossing a coin

...	$i = 5$	$i = 6$	$i = 7$	...
...	approx T	approx H	approx H	...

With $\pi = 0.5$,

the probability of "approximately HEAD" is

$$\begin{aligned}\mathbb{P}_X(X \stackrel{\sim}{=} 0) &= 0.9 \cdot (0.5^0 \cdot (1 - 0.5)^1) + 0.2 \cdot (0.5^1 \cdot (1 - 0.5)^0) \\ &= 0.9 \cdot 0.5 + 0.2 \cdot 0.5 = 0.55\end{aligned}$$

whereas the probability of "approximately TAIL" is

$$\begin{aligned}\mathbb{P}_X(X \stackrel{\sim}{=} 1) &= 0.1 \cdot (0.5^0 \cdot (1 - 0.5)^1) + 0.8 \cdot (0.5^1 \cdot (1 - 0.5)^0) \\ &= 0.1 \cdot 0.5 + 0.8 \cdot 0.5 = 0.45\end{aligned}$$

Monotone measures
Possibility, Probability, Beliefs..

Modeling fuzziness as source of uncertainty, requires introducing the notion of **monotone measures** [27] which allows for representing fuzziness in terms of **possibility measure**.

Broadly speaking, fuzzy membership function $\xi_{\widetilde{X}}$ can be interpreted as possibility measure [12], which in turn belongs to the family of **imprecise probabilities** (in particular, *p-boxes*).

For further details, refer to [1].

Fuzzy probability
Fuzzy random variables

Since Zadeh's definition of probability of a fuzzy event [37], there have been a plethora of attempts to combine fuzziness and randomness (e.g., see [3, 38, 18, 33, 1, 8, 26, 6]), often diverging from one another.

Limitations: The general **lack** of operational parametric probability models for fuzzy random variables, along with the absence of a **unified inferential framework** that provides widely applicable asymptotic results, makes these models **challenging to apply**.

Fuzzy probability

Fuzzy random variables

A **fuzzy random variable** is a random variable whose possible outcomes are fuzzy numbers instead of real numbers.

There are two main approaches:

- **Ontic**: The focus is on the <u>non-standard</u> random mechanism that produces fuzzy outcomes. In this case, the phenomenon being modeled is fuzzy ontologically (i.e., *fuzzy in nature*).

Fuzzy probability

Fuzzy random variables

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There are two main approaches:

- **Ontic**: The focus is on the <u>non-standard</u> random mechanism that produces fuzzy outcomes. In this case, the phenomenon being modeled is fuzzy ontologically (i.e., *fuzzy in nature*).

- **Epistemic**: Fuzzy numbers represent descriptions of unobserved (latent) crisp random variables. In this case, the phenomenon being modeled is fuzzy because of a lack of ability/knowledge in observing the true but latent outcomes (i.e., *fuzzy in knowledge*).

Fuzzy probability

Fuzzy random variables

A **fuzzy random variable** is a random variable whose possible outcomes are fuzzy numbers instead of real numbers.

There are two main approaches:

- **Ontic**: The focus is on the <u>non-standard</u> random mechanism that produces fuzzy outcomes. In this case, the phenomenon being modeled is fuzzy ontologically (i.e., *fuzzy in nature*).

- **Epistemic**: Fuzzy numbers represent descriptions of unobserved (latent) crisp random variables. In this case, the phenomenon being modeled is fuzzy because of a lack of ability/knowledge in observing the true but latent outcomes (i.e., *fuzzy in knowledge*).

  $\rightarrow$ Related to **censored** and **coarse data** [5, 31]

Fuzzy probability

Fuzzy random variables (Ontic approach)

Let the family of real fuzzy numbers be

$$\mathcal{F}(\mathbb{R}) = \{\xi_{\tilde{A}} : \mathbb{R} \to [0,1] \mid \xi_{\tilde{A}_\alpha} \in \mathcal{K}(\mathbb{R}), \alpha \in [0,1]\}$$

with $\mathcal{K}(\mathbb{R})$ being the family of all non-empty compact intervals.

Consider two fuzzy numbers $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$, then:

- the α cut-based operations of summation $\tilde{A} + \tilde{B}$ (Minkowski'sum) and (scalar) product $b\tilde{A}$ are applied element-wise on the elements of intervals
- the difference $\tilde{A} -_H \tilde{B}$ is the Hukuhara difference
- $D_\tau^\lambda(\tilde{A}, \tilde{B})$ is used to measure distances

Fuzzy probability
Fuzzy random variables (Ontic approach)

A fuzzy random variable is a Borel measurable mapping $\tilde{X}$ from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the metric space $(\mathcal{F}(\mathbb{R}), D_\tau^\lambda)$.

Note: given the α-cuts representation, $\tilde{X}$ works level-wise.

Given a collection of fuzzy rvs $\mathcal{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$:

- $\tilde{\mu} = \mathbb{E}[\mathcal{X}] = \left[\mathbb{E}[\text{mid}\mathcal{X}_\alpha] - \mathbb{E}[\text{spr}\mathcal{X}_\alpha], \mathbb{E}[\text{mid}\mathcal{X}_\alpha] + \mathbb{E}[\text{spr}\mathcal{X}_\alpha]\right] \in \mathcal{F}(\mathbb{R})$
- $\sigma^2_{\mathcal{X}} = \mathbb{V}\text{ar}[\mathcal{X}] = \mathbb{E}\left[D_\tau^\lambda(\mathcal{X}, \tilde{\mu})^2\right] \in \mathbb{R}^+$
- If $\mathcal{Y}$ is also available:
  $\sigma_{\mathcal{X},\mathcal{Y}} = \mathbb{C}\text{ov}[\mathcal{X}, \mathcal{Y}]_{|D_\tau^\lambda} = \mathbb{C}\text{ov}[\text{mid}\mathcal{X}, \text{mid}\mathcal{Y}] + \mathbb{C}\text{ov}[\text{spr}\mathcal{X}, \text{spr}\mathcal{Y}] \in \mathbb{R}$

Fuzzy probability
Fuzzy random variables (Ontic approach)

Note:

- $\tilde{\mu}$ preserves all the main properties from the crisp case (e.g., additivity, equivariance under translation and product by a scalar)
- $\sigma^2_{\mathcal{X}}$ preserves all the main properties from the crisp case (e.g., it vanishes with degenerate distribution, invariance under translation, additivity under independence)
- $\sigma_{\mathcal{X},\mathcal{Y}}$ preserves some properties from the crisp case (e.g., it vanishes for independent random variables)

Fuzzy probability
Fuzzy random variables (Ontic approach)

Some limitations of the Ontic approach if compared to standard statistics:

- Use of Hukuhara operator to approximate difference between fuzzy numbers
- Lack of a general total ranking between fuzzy numbers
- No flexible models for fuzzy random variables
- Limit theorems do not always apply for fuzzy random variables (lack of easy-to-apply statistical inference)

Outline

Overview

Regardless of both epistemic and ontic approaches to statistics, we can recognize that fuzzy numbers allow for representing a systematic and non-random uncertainty associated with *data*, *parameters*, or *hypotheses*.

Overview

Consider a parametric statistical model of the form

$$\mathcal{M} = \{F(\mathbf{y};\boldsymbol{\theta}), \boxed{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \subset \mathbb{R}^p, \boxed{\mathbf{y} \in \mathcal{Y}^n}\}$$

Fuzzy numbers can be introduced in this context to:

- represent the **parameter space** $\boldsymbol{\Theta}$
- represent the **sample space** $\mathcal{Y}^n$

In both cases, the statistical model needs to deal with two different sources of uncertainty at least: the **aleatoric uncertainty** and the **fuzziness** (provided by fuzzy numbers).

Overview

Fuzziness in $\boldsymbol{\Theta}$

When the parameter space is a subset of fuzzy numbers, i.e. $\boldsymbol{\Theta} \subset \mathcal{F}(\mathbb{R})^p$, we may have several statistical models where **parameters are fuzzy sets**:

- Fuzzy clustering methods (e.g., [11])
- Fuzzy regression (e.g., [7])
- Fuzzy time series models (e.g., [7])
- ...

Overview

Fuzziness in $\mathcal{Y}^n$

When the sample space is a subset of fuzzy numbers, e.g. $\mathcal{Y}^n \subset \mathcal{F}(\mathbb{R})^n$ or $\mathcal{Y}^n \subset \mathcal{F}(\mathbb{N})^n$, we may have several statistical models where **data are fuzzy sets**:

- Fuzzy linear models (e.g., [4, 7])
- Fuzzy time series models (e.g., [7])
- Fuzzy component analysis (e.g., [20])
- ...

In these cases, sufficient statistics for estimating model parameters are based on fuzzy data as well. The inferential mechanism has to be generalized according to this new type of data.

Overview
Fuzzy hypothesis testing

Statistical hypothesis testing can also be performed using a set of *fuzzy hypotheses* over $\boldsymbol{\Theta}$:

- Fuzzy Neyman-Pearson lemma for fuzzy-(U)MP test [34, 36]
- Fuzzy p-values [14, 17]
- Fuzzy Bayesian inference [15, 17]

Overview
What links fuzzy to traditional statistics?

Fuzzy statistics shares some commonalities with more traditional approaches dealing with

- masked data [30]
- symbolic data analysis [2]
- interval-censored data [29]
- coarse data [25, 32]

Overview
What links fuzzy to traditional statistics?

Adopting the perspective of **coarse data** enables the adoption of the so-called **epistemic view** on fuzzy statistics [8].

Consequently, fuzzy numbers would represent a **blurred version** of unobserved crisp random variables. The modeled phenomenon is considered fuzzy due to a lack of ability to observe the true outcomes (i.e., *fuzziness in knowledge*).

Overview
What links fuzzy to traditional statistics?

The sampling process is seen as a **two-stage mechanism**: first, a random experiment, and then the fuzzification of the observed outcomes.

Overview
What links fuzzy to traditional statistics?

Imagine you are a professional chocolate taster and consider the case of classifying chocolate types, which vary based on **aromatic compounds** (e.g., pyrazine, aldehyde), based on their **aroma intensity** using a **fuzzy variable** with four categories (e.g., *intenso*, *avvolgente*, *deciso*, and *prodigioso*).

Overview
What links fuzzy to traditional statistics?

The simple process of counting how many chocolate bars are of a particular aroma intensity gives rise to **fuzzy counts**.

This results in a collection of (fuzzy) natural numbers that reflect the inherent variability in chocolate types and the subjective nature of aroma intensity evaluation.

Fuzzy regression
Maximum Likelihood regression with fuzzy data

Under the previous epistemic scenario, consider the simplest case of the Normal linear model:

$$\mathcal{M} = \left\{\mathcal{N}(\mathbf{y};\boldsymbol{\mu},\mathbf{I}\sigma^2), \boldsymbol{\theta} = \{\boldsymbol{\mu},\sigma^2\} \subset \mathbb{R}^n \times \mathbb{R}^+, \mathbf{y} \in \mathbb{R}^n\right\}$$

where the linear predictor (mean) of the model is a linear combination of p predictors $\mathbf{X}_{n\times p}$:

$$\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$$

As usual, the interest lies in identifying the true model $\mathcal{M}^0$ which has generated the sample data $\mathbf{y}$ (i.e., estimate the unknown parameters $\{\boldsymbol{\beta},\sigma^2\}$).

Fuzzy regression
Maximum Likelihood regression with fuzzy data

Now, assuming that $\mathbf{y}$ cannot be observed directly due to non-random sources of uncertainty (*post-sampling errors*) but the fuzzy sample $\tilde{\mathbf{y}}$ is instead available.

The parameters $\{\boldsymbol{\beta},\sigma^2\}$ of the Normal linear model with fuzzy observations can be estimated, for instance, using *maximum likelihood* adapted to deal with fuzzy numbers [19].

Fuzzy regression
Maximum Likelihood regression with fuzzy data

The **fuzzy likelihood function** is as follows:

$$\mathcal{L}(\boldsymbol{\beta}, \sigma^2; \tilde{\mathbf{y}}) = \prod_{i=1}^{n} \int_{\text{supp}(\tilde{y}_i)} \xi_{\tilde{y}_i}(y) \mathcal{N}(y; \boldsymbol{\beta}, \sigma^2)\, dy$$

where $(\xi_{\tilde{y}_1}, \ldots, \xi_{\tilde{y}_i}, \ldots, \xi_{\tilde{y}_n})$ is the sample of fuzzy data (e.g., trapezoidal fuzzy numbers).

Fuzzy regression
Maximum Likelihood regression with fuzzy data

$$\mathcal{L}(\boldsymbol{\beta}, \sigma^2; \tilde{\mathbf{y}}) = \prod_{i=1}^{n} \int_{\text{supp}(\tilde{y}_i)} \xi_{\tilde{y}_i}(y) \mathcal{N}(y; \boldsymbol{\beta}, \sigma^2)\, dy$$

fuzziness of data through fuzzy numbers

Antonio Calcagnì $2GM/c^2$
May 2025 Fuzzy linear regression 40/57

Fuzzy regression
Maximum Likelihood regression with fuzzy data

$$\mathcal{L}(\boldsymbol{\beta}, \sigma^2; \tilde{\mathbf{y}}) = \prod_{i=1}^{n} \int_{\text{supp}(\tilde{y}_i)} \xi_{\tilde{y}_i}(y) \mathcal{N}(y; \boldsymbol{\beta}, \sigma^2)\, dy$$

fuzziness of data through fuzzy numbers

randomness of data through the probabilistic model

Antonio Calcagnì $2GM/c^2$
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Fuzzy regression
Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

Given a candidate $\boldsymbol{\theta}'$, the EM algorithm iterates between:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right]$
- **M-step**
  Maximize $\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y})$ by replacing $\mathbf{y}$ with $\mathbf{y}^*$

Fuzzy regression

Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right]$

  → Likelihood of the Normal linear model with no fuzzy observations

Fuzzy regression

Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right]$

  → Likelihood of the Normal linear model with no fuzzy observations

  → filtered data: what we would expect to observe if fuzziness was not present in the data

Fuzzy regression

Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right] =$

$$= \int y \, \frac{\xi_{\tilde{y}_i}(y)\mathcal{N}(y; \boldsymbol{\theta}')}{\int \xi_{\tilde{y}_i}(z)\mathcal{N}(y; \boldsymbol{\theta}') \, dz} \, dy$$

Fuzzy regression

Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right] =$

$$= \int y \, \frac{\xi_{\tilde{y}_i}(y)\mathcal{N}(y; \boldsymbol{\theta}')}{\int \xi_{\tilde{y}_i}(z)\mathcal{N}(y; \boldsymbol{\theta}') \, dz} \, dy$$

  → Normal density function *conditioned* on fuzzy numbers!

Fuzzy regression
Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right] =$

$$= \int y \, \frac{\xi_{\tilde{y}_i}(y)\mathcal{N}(y;\boldsymbol{\theta}')}{\underbrace{\int \xi_{\tilde{y}_i}(z)\mathcal{N}(y;\boldsymbol{\theta}')\, dz}_{\text{normalization constant}}} \, dy$$

Fuzzy regression
Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **E-step**
  Compute $\mathbf{y}^* = \mathbb{E}_{\boldsymbol{\theta}'}\left[\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y}) | \tilde{\mathbf{y}}\right] =$

$$= \int y \, \frac{\overbrace{\xi_{\tilde{y}_i}(y)}^{\text{fuzziness}}\;\overbrace{\mathcal{N}(y;\boldsymbol{\theta}')}^{\text{randomness}}}{\underbrace{\int \xi_{\tilde{y}_i}(z)\mathcal{N}(y;\boldsymbol{\theta}')\, dz}_{\text{normalization constant}}} \, dy$$

Fuzzy regression
Maximum Likelihood regression with fuzzy data

The parameters $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \sigma^2\}$ can be estimated via Expectation Maximization (EM) [9]:

- **M-step**
  Maximize $\ln \mathcal{L}(\boldsymbol{\beta}, \sigma^2; \mathbf{y})$ by replacing $\mathbf{y}$ with $\mathbf{y}^*$:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}^*$$

$$\hat{\sigma}^2 = \frac{1}{n}\left(\mathbf{y}^* - \mathbf{X}\hat{\boldsymbol{\beta}}\right)^T\left(\mathbf{y}^* - \mathbf{X}\hat{\boldsymbol{\beta}}\right)$$

  where $\mathbf{y}^*$ are the observations filtered from fuzziness.

Fuzzy regression

Without modeling the fuzzy sampling process, one might solve the fuzzy regression problem using an optimization approach. In the literature, there have been thousands of proposals [7], all of them revolving around the following generalizations:

(a) $\mathbf{y} = \mathbf{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{\epsilon}$ (Crisp response, crisp predictors, fuzzy coefficients)
(b) $\tilde{\mathbf{y}} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ (Fuzzy response, crisp predictors, crisp coefficients)
(c) $\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ (Fuzzy response, fuzzy predictors, crisp coefficients)
(d) $\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}} + \boldsymbol{\epsilon}$ (Fuzzy response, fuzzy predictors, fuzzy coefficients)

Fuzzy regression
Possibilistic regression (Tanaka's approach)

The simplest formulation for $\mathbf{y} = \mathbf{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{\epsilon}$ has been provided by Tanaka [35]:

$$\begin{aligned} y_i &= \tilde{\beta}_0 + \sum_j x_{ij}\tilde{\beta}_j + \epsilon_i \\ &= (m^{\beta_0}, s^{\beta_0}) + \sum_j x_{ij}(m_j^{\beta_j}, s_j^{\beta_j}) + \epsilon_i \end{aligned}$$

where the regression coefficients have been expressed as *symmetric triangular fuzzy numbers*, i.e. $\tilde{\beta} = (m, s)_{LR}$.

Note that the linear model predicts outcomes in terms of fuzzy numbers, i.e. $\hat{\mathbf{y}} = \hat{\tilde{\mathbf{y}}}$.

Fuzzy regression
Possibilistic regression (Tanaka's approach)

The regression model is determined by minimizing the *sum of spreads* of the estimated fuzzy outputs:

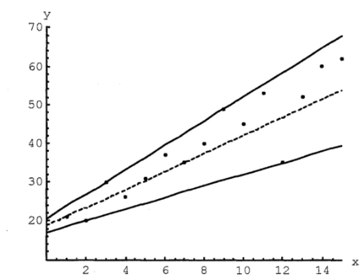
$$\min_{\mathbf{m},\mathbf{s}} \sum_i \sum_j |x_{ij}| c_j$$

$$\text{subject to:} \quad {\scriptstyle (1)}\ s_j \geq 0 \quad {\scriptstyle (2)}\ y_i \in \mathcal{S}(\hat{\tilde{y}}_i)_h \quad i = 1, \ldots, n$$

where cons. (2) indicates that the predicted outcomes $\hat{\tilde{\mathbf{y}}}$ have to lie inside the h-level sets of the estimated regression lines.

Fuzzy regression
Possibilistic regression (Tanaka's approach)

No.(j)	1	2	3	4	5	6	7	8
x	1	2	3	4	5	6	7	8
y	21	20	30	26	31	37	35	40
No.(j)	9	10	11	12	13	14	15	
x	9	10	11	12	13	14	15	
y	49	45	53	35	52	60	62	



Source: [28]

Fuzzy regression
Possibilistic regression (Tanaka's approach)

Generalization of the Tanaka's proposal have been made so as to include asymmetric fuzzy numbers, fuzzy outcome and predictors. For further details, see: [28].

Fuzzy regression

Least squares regression with fuzzy data

Consider a sample of fuzzy triangular numbers $\tilde{\mathbf{y}} = ((m_1, l_1, r_1), \ldots, (m_n, l_n, r_n))_{LR}$ and a matrix of crisp predictors $\mathbf{X}_{n \times p}$. Then, given the LR parametric representation, the following (*non-interactive*) linear model can be formulated [13]:

$$\begin{aligned}
\mathbf{m} &= \mathbf{X}\boldsymbol{\beta}_m + \boldsymbol{\epsilon}_m \\
\mathbf{l} &= \mathbf{X}\boldsymbol{\beta}_l + \boldsymbol{\epsilon}_l \\
\mathbf{r} &= \mathbf{X}\boldsymbol{\beta}_r + \boldsymbol{\epsilon}_r
\end{aligned}$$

Fuzzy regression

Least squares regression with fuzzy data

Alternatively, a (*interactive*) linear model could also be developed:

$$\begin{aligned}
\mathbf{m} &= \overbrace{\mathbf{X}\boldsymbol{\beta}_m}^{\mathbf{m}^*} + \boldsymbol{\epsilon}_m \\
\mathbf{l} &= \mathbf{m}^*\beta_l + \boldsymbol{\epsilon}_l \\
\mathbf{r} &= \mathbf{m}^*\beta_r + \boldsymbol{\epsilon}_r
\end{aligned}$$

Fuzzy regression

Least squares regression with fuzzy data

In both cases, the regression problem is solved by formulating the following (unconstrained) problem:

$$\min_{\boldsymbol{\beta}} \quad \|\mathbf{m} - \mathbf{m}^*\|_2^2 + \|\mathbf{l} - \mathbf{l}^*\|_2^2 + \|\mathbf{r} - \mathbf{r}^*\|_2^2$$

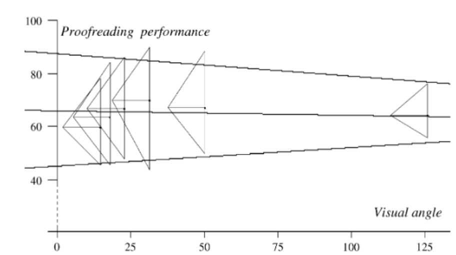
which can be constrained by letting the spread components to be non-negative:

$$\mathbf{l}^* \geq \mathbf{0}_n \qquad \text{and} \qquad \mathbf{r}^* \geq \mathbf{0}_n$$

Fuzzy regression

Least squares regression with fuzzy data

Dataset		Parameter estimates
X	$Y \equiv (c, p, q)$	
14.8	(59.7, 13.7, 18.3)	
18.0	(63.5, 17.5, 20.5)	$\hat{\mathbf{a}} = (65.940128, -0.016853)'$
22.9	(66.8, 18.8, 19.2)	$\hat{b} = 5.1076942$
31.5	(70, 26, 20)	$\hat{d} = -316.155$
50.3	(67, 17, 21)	$\hat{g} = 3.9716534$
126.0	(64.2, 8.2, 11.8)	$\hat{h} = -240.4851$



Source: [13]

Fuzzy regression
Least squares regression with fuzzy data

Generalization of the least squares fuzzy regression have been made so as to include trapezoidal fuzzy numbers, crisp outcome and fuzzy predictors.
For further details, see: [13].

Fuzzy bootstrap
Overview

Fuzzy random variables constitute a well-founded approach to deal with fuzziness in statistical data analysis. However, due to the limitations on asymptotic results useful for doing inference, inference with fuzzy rvs is still an open issue.

To overcome some of the current limitations, bootstrap techniques have been widely adopted in fuzzy statistics (e.g., [21]).

Two recent proposals will be briefly considered here:

- Bootstrap for *epistemic* fuzzy data [23]
- Bootstrap for *ontic* fuzzy data [22]

Fuzzy bootstrap
Epistemic approach

Suppose a sample $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ of fuzzy observations (e.g., trapezoidal/triangular) is available given a collection of (latent/unobserved) crisp random variables $(X_1, \ldots, X_n)$. In this context, $\tilde{x}_i \in \mathcal{F}(\mathbb{R})$ and $[\tilde{x}_i]_\alpha$ denotes the α-cut of $\tilde{x}_i$ (bounded interval).

Then, for $b = 1, \ldots, B$ [22]:

s_1 : $\alpha \sim \mathcal{U}(;0,1)$

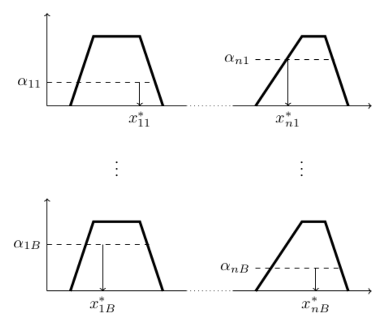
s_2 : $\hat{x}_i^{(b)} \sim \mathcal{U}(;\min\,[\tilde{x}_i]_\alpha, \max\,[\tilde{x}_i]_\alpha)$

Do s_1-s_2 for $i = 1, \ldots, n$

The bootstrap sample $(\hat{x}_1, \ldots, \hat{x}_n)^{(b)}$ constitutes a random sample from $(X_1, \ldots, X_n)$ which can be used to compute statistics of interest $T_B(\hat{\mathbf{x}}^{(b)})$ as usual.

Fuzzy bootstrap
Epistemic approach

Graphical representation of steps s_1 and s_2 for trapezoidal fuzzy numbers:



Source: [22]

Fuzzy bootstrap
Epistemic approach

To reduce the variance of the bootstrap-based estimates, well-known Monte Carlo techniques (e.g., the *antithetic variates method*) can be generalized as well. For further details, see [22].

Fuzzy bootstrap
Ontic approach

Let $\mathcal{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ be a collection of fuzzy rvs with $\tilde{\mathbf{x}}$ being a sample of n trapezoidal fuzzy numbers.

The i-th observation is parameterized as $\tilde{x}_i = (m, s, l, r)$, with m and s denoting the center and the width of the core, l and r denoting the usual left and right spreads.

The following quantities can be used to synthesize $\tilde{x}_i$ (*canonical representation*) [23]:

- $\text{Val}(\tilde{x}) = c + (r - l)\frac{1}{6}$ (Location of the fuzzy number)
- $\text{Amb}_L(\tilde{x}) = \frac{1}{2}s + \frac{1}{6}l$ (Left ambiguity of the fuzzy number)
- $\text{Amb}_U(\tilde{x}) = \frac{1}{2}s + \frac{1}{6}r$ (Right ambiguity of the fuzzy number)
- $\text{EV}(\tilde{x}) = c + (r - l)\frac{1}{4}$ (Exp. value of the fuzzy number)
- $\text{w}(\tilde{x}) = s + (r + l)\frac{1}{4}$ (Width of the fuzzy number)

Fuzzy bootstrap
Ontic approach

In this case, the bootstrap technique can be applied on the canonical representation of the fuzzy sample $\tilde{\mathbf{x}}$. Note that, although the fuzzy observations do not need to obey a particular shape, the bootstrap samples are always represented in terms of trapezoidal fuzzy numbers.

Several algorithms can be defined based on preserving some properties of the canonical representation, e.g. **VA**, **VAA**, **VAF**.

Fuzzy bootstrap
Ontic approach

For the sake of simplicity, the **VAA algorithm** is reproduced here.

Require: Fuzzy sample $x_1, \ldots, x_n \in \mathbb{F}(\mathbb{R})$
Ensure: A flexible bootstrap sample
1: **for** $i = 1$ to n **do**
2: Compute $\text{Val}(x_i), \text{Amb}_L(x_i), \text{Amb}_U(x_i)$
3: **end for**
4: **for** $i = 1$ to n **do**
5: Generate randomly (with equal probabilities) a triple $(\text{Val}^*, \text{Amb}_L^*, \text{Amb}_U^*)$ from

$$\left\{\left(\text{Val}(x_1), \text{Amb}_L(x_1), \text{Amb}_U(x_1)\right), \ldots, \left(\text{Val}(x_n), \text{Amb}_L(x_n), \text{Amb}_U(x_n)\right)\right\}$$

6: $c^* \leftarrow \text{Val}^* + \text{Amb}_U^* - \text{Amb}_L^*$
7: Generate s^* from the uniform distribution on the interval

$$\left[0, 2\min\left\{\text{Amb}_L^*, \text{Amb}_U^*\right\}\right]$$

8: $l^* \leftarrow 6\text{Amb}_L^* - 3s^*$
9: $r^* \leftarrow 6\text{Amb}_U^* - 3s^*$
10: $x_i^* \leftarrow x_i^*(c^*, s^*; l^*, r^*)$
11: **end for**

Source: [23]

Outline

Concluding remarks

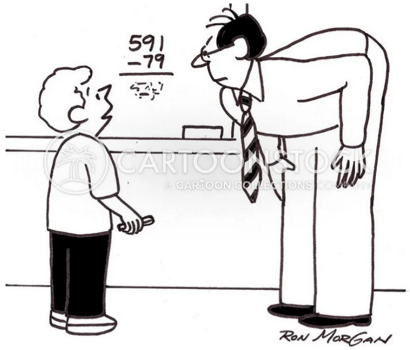
- Statistical modeling with fuzzy numbers can be of relevant importance in all those situations involving **non-standard uncertainty** (e.g., epistemic)
- A consistent statistical modeling framework to deal with **fuzzy data analysis** is necessary for modelers and data analysts (i.e., general regression-like approach)

Concluding remarks

- However, some rigorously probabilistic proposals are **far** from being **widely applicable**: statistical models, estimators, and inferential properties need to be constructed and verified each time
- Some working proposals are entirely probabilistic: **fuzziness is summarized** into (a few) statistics sacrificing its full expressiveness

Concluding remarks

- After 60 years of fuzzy research, we should ask to what extent fuzzy mathematics remains **relevant today**, especially when modeling imprecise data, and whether its goals can be incorporated into **existing statistical approaches** (e.g., Bayesian Networks)



"It's fuzzy math."

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